

# NON-FIBERED L-SPACE KNOTS

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**ABSTRACT.** We construct an infinite family of knots in rational homology spheres with irreducible, non-fibered complements, for which every non-longitudinal filling is an L-space.

## 1. INTRODUCTION

The Heegaard Floer homology of a three-manifold  $Y$  is an abelian group  $\widehat{HF}(Y)$  satisfying  $\text{rk } \widehat{HF}(Y) \geq |H_1(Y; \mathbb{Z})|$ . When equality is realized in this bound,  $Y$  is called an L-space, and any knot in  $Y$  admitting a non-trivial L-space surgery is called an L-space knot. A result of Ghiggini [6] and Ni [9] shows that L-space knots in the three-sphere must be fibered. Since manifolds with finite fundamental group provide examples of L-spaces,<sup>1</sup> this result implies that a knot  $K$  in  $S^3$  admitting a finite filling must be fibered. This observation should be compared with other restrictions related to finite fillings such as the Cyclic Surgery Theorem [5] and its extensions [4].

The restriction to knots in  $S^3$  is not necessary: it is shown in [1] that a primitive knot in an irreducible L-space admitting a non-trivial L-space surgery must be fibered. Irreducibility is required, as removing an unknot from an embedded three-ball in any L-space produces a non-fibered manifold with non-trivial L-space fillings. However, in the general setting of knots in rational homology spheres with irreducible complements fibered is not a necessary condition:

**Theorem 1.** *There exist infinitely many irreducible, non-fibered knot complements such that all non-longitudinal Dehn fillings are L-spaces. Moreover, these examples arise as knots in manifolds with finite fundamental group.*

## 2. PRELIMINARIES

We begin by fixing terminology. Fibrations will always be locally trivial surface bundles over a circle and we say the total space fibers. To avoid confusion, we will refer to Seifert fibrations as Seifert structures; these are foliations of a manifold by circles. The base orbifold is the leaf space of such a foliation, where the (possibly empty) collection of cone points records the multiplicities of the exceptional fibers in the Seifert structure. A circle bundle is a Seifert structure for which there are no exceptional fibers.

Given a slope  $\alpha$  on a manifold  $M$  with torus boundary, we use  $M(\alpha)$  to denote Dehn filling along  $\alpha$ . If  $\partial M = T_1 \cup T_2$ , for tori  $T_i$ , then we denote  $\alpha$ -filling on  $T_1$  (respectively  $T_2$ ) by  $M(\alpha, -)$  (respectively  $M(-, \alpha)$ ). When  $M$  admits a Seifert structure, the slope given by a regular fiber in the boundary is called the fiber slope. For background on Seifert structures and Dehn filling we refer the reader to Boyer [2]. A key fact is that Dehn filling a Seifert manifold with torus boundary along any slope  $\alpha$  other than the fiber slope results in a Seifert manifold with an additional singular fiber. The multiplicity of this new fiber is  $\Delta(\alpha, \varphi)$ , the distance between the slopes  $\alpha$  and  $\varphi$  [7].

In general, we will consider oriented manifolds  $M$  with torus boundary for which  $H_1(M; \mathbb{Q}) \cong \mathbb{Q}$ . These arise as the complements of knots in rational homology spheres. As such, there is always a

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<sup>1</sup>Ozsváth and Szabó show that manifolds admitting elliptic geometry are L-spaces [10]; the Geometrization Theorem [8] implies that three-manifolds with finite fundamental group admit elliptic geometry.

preferred slope given by the rational longitude, characterized by the property that some number of like-oriented parallel copies bounds a properly embedded surface in  $M$ . We will refer to this slope as the longitude.

Let  $N$  denote the twisted  $I$ -bundle over the Klein bottle; a Heegaard diagram for this manifold with torus boundary is given in Figure 1. As there is a unique line bundle over the Klein bottle with oriented total space, the manifold  $N$  is unique. This manifold can be given two different Seifert structures. The first is by treating  $N$  as a circle bundle over the Möbius band. We denote the fiber slope in this Seifert structure by  $\phi_0$ . This slope coincides with the longitude of  $N$ , and this circle bundle gives  $N$  the structure of an annulus fibration over the circle. The other Seifert structure has base orbifold  $D^2(2, 2)$  (a disk with two cone points each of order 2); the fiber slope here is denoted  $\phi_1$ . These conventions are consistent with [3, Section 3]. It can be shown that  $\Delta(\phi_0, \phi_1) = 1$  and any filling  $N(\alpha)$  for which  $\alpha \neq \phi_0, \phi_1$  admits a pair of Seifert structures with base orbifolds  $\mathbb{R}P^2(\Delta(\alpha, \phi_0))$  and  $S^2(2, 2, \Delta(\alpha, \phi_1))$ . We point out that these manifolds always admit elliptic geometry [11].

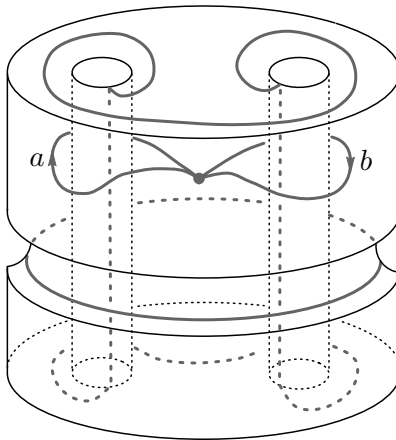


FIGURE 1. A Heegaard diagram for the twisted  $I$ -bundle over the Klein bottle  $N$ . With  $a$  and  $b$  generating the fundamental group of the genus two handlebody,  $N$  is obtained by attaching a handle along a curve in the boundary representing  $a^2b^2$  so that  $\phi_0 \simeq ab$  and  $\phi_1 \simeq b^2$ . An annulus in the boundary with core representing the element  $\phi_0 \simeq ab$  may be used to find the fundamental group of  $M$ , the complement of a regular fiber in the interior of  $N$ , via HNN extension.

Since  $N$  is homotopy equivalent to a Klein bottle, we have  $\pi_1(N) = \langle a, b | a^2b^2 \rangle$ . Note that this presentation may be easily deduced from the Heegaard diagram in Figure 1. The longitude of  $N$  is homotopic to the element  $ab$  (this element has order two in the abelianization of  $\pi_1(N)$ ).

Now consider a knot  $K_0$  in  $N$  that is isotopic to  $\phi_0$ . Define  $M$  by removing a neighborhood of  $K_0$  from  $N$ ; by construction  $M$  inherits a Seifert structure (the base orbifold is a punctured Möbius band). Now  $\partial M = T_1 \cup T_2$  where  $T_2$  denotes the boundary of a regular neighborhood of  $K_0$ .

The fundamental group of  $M$  is presented by

$$\pi_1(M) = \langle a, b, t | a^2b^2, [t, ab] \rangle.$$

To see this, consult Figure 1 and notice that  $M$  may be constructed by identifying (disjoint neighborhoods of) each boundary component of the annulus with core  $ab$  in  $\partial N$ . This gives rise to the HNN extension presented above. Notice that  $M(-, \mu) \cong N$  for any slope satisfying  $\Delta(\mu, \phi_0) = 1$ . A preferred choice for  $\mu$  is given by a representative of the homotopy class of  $t$  in the above presentation. Notice that  $\phi_0$ , as a regular fiber in  $M$ , also gives a slope on  $T_2$ . Using a self-diffeomorphism of  $M$  which exchanges  $T_1$  and  $T_2$ ,  $M(\alpha, -)$  is also homeomorphic to  $N$  if  $\Delta(\alpha, \phi_0) = 1$ .

Our interest is in the family of manifolds  $M(-, \alpha)$  for any slope  $\alpha$  with  $\Delta(\alpha, \phi_0) > 1$ . Notice that each of these manifolds admits a Seifert structure with base orbifold a Möbius band with a single cone point of order  $\Delta(\alpha, \phi_0)$ . Since  $M(\phi_1, \alpha)$  admits a Seifert structure with base orbifold  $S^2(2, 2, n)$  it follows that  $M(-, \alpha)$  is the complement of a knot in an elliptic manifold for all  $\alpha$ .

### 3. THE PROOF OF THEOREM 1

**Lemma 2.** *Fix a slope  $\alpha$  on  $T_2$  with  $\Delta(\alpha, \phi_0) = p$ . Then*

$$M(\phi_0, \alpha) = \begin{cases} S^2 \times S^1 \# S^2 \times S^1 & \text{if } p = 0, \\ S^2 \times S^1 \# L(p, q) & \text{if } p > 1, \\ S^2 \times S^1 & \text{if } p = 1. \end{cases}$$

*Proof.* Since

$$\pi_1(M) \cong \langle a, b, t | a^2 b^2, [t, ab] \rangle$$

and  $\phi_0 \simeq ab$ , we have that

$$\begin{aligned} \pi_1(M(\phi_0, -)) &\cong \langle a, b, t | a^2 b^2, [t, ab] \rangle / \langle \langle ab \rangle \rangle \\ &\cong \langle a, b, t | ab \rangle. \end{aligned}$$

In other words,  $\pi_1(M(\phi_0, -)) \cong \mathbb{Z} * \mathbb{Z}$ . If  $\alpha = p\mu + q\phi_0$ , then

$$\begin{aligned} \pi_1(M(\phi_0, \alpha)) &\cong \langle a, b, t | ab \rangle / \langle \langle t^p (ab)^q \rangle \rangle \\ &\cong \mathbb{Z} * \mathbb{Z}/p. \end{aligned}$$

By Whitehead's proof of Kneser's conjecture [12],  $M(\phi_0, \alpha)$  is a connect-sum of closed manifolds  $Y_1$  and  $Y_2$  with  $\pi_1(Y_1) \cong \mathbb{Z}$  and  $\pi_1(Y_2) \cong \mathbb{Z}/p$ . Geometrization now establishes the lemma.  $\square$

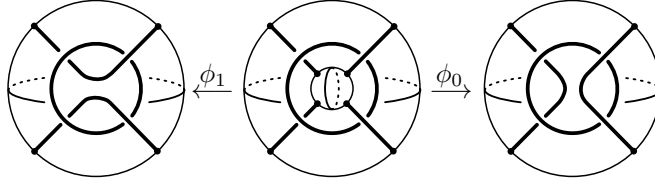


FIGURE 2. The branch set for the manifold  $M = M(-, -)$  with branch sets for the fillings  $N = M(\phi_1, -)$  and  $M(\phi_0, -)$ . Notice that the latter manifold is reducible, containing an  $S^2 \times S^1$  summand.

*Remark 3.* Alternatively, Lemma 2 follows from considering  $M(\phi_0, -)$  as the double branched cover of a tangle as in Figure 2. The unknotted component gives rise to the  $S^2 \times S^1$  summand. Dehn filling corresponds to attaching a rational tangle, which (ignoring the unknotted component) produces a two-bridge link and exhibits the lens space connect-summand.

**Proposition 4.** *For any  $\alpha$  on  $T_2$  with  $\Delta(\alpha, \phi_0) \neq 1$ , the manifold  $M(-, \alpha)$  does not fiber.*

*Proof.* Suppose that  $M(-, \alpha)$  fibers. Since  $\phi_0$  is the longitude, this is the only filling that extends the fibration on  $M(-, \alpha)$ , as any other filling of  $M(-, \alpha)$  results in a rational homology sphere. By Lemma 2,  $M(\phi_0, \alpha) \cong S^2 \times S^1 \# L(p, q)$  for  $p = \Delta(\phi_0, \alpha) \geq 2$ , and  $M(\phi_0, \phi_0) \cong S^2 \times S^1 \# S^2 \times S^1$ . Because  $M(\phi_0, \alpha)$  is fibered and  $\pi_2(M(\phi_0, \alpha)) \neq 0$ , the fiber surface  $F$  must also have  $\pi_2(F) \neq 0$ , and hence  $F$  must be  $S^2$  or  $\mathbb{R}P^2$ . However,  $\pi_1(M(\phi_0, \alpha))$  is not the fundamental group of such a fibration, since it does not admit a surjective homomorphism onto  $\mathbb{Z}$  with finite kernel.  $\square$

*Proof of Theorem 1.* Fix  $\alpha$  with  $\Delta(\alpha, \phi_0) \geq 2$ . As the fiber slope of the Seifert structure on  $M(-, \alpha)$  is the longitude, all non-longitudinal fillings will extend the Seifert structure, yielding a base orbifold  $\mathbb{R}P^2$  with two cone points. By [3, Proposition 5], such manifolds are always L-spaces. Proposition 4 shows that  $M(-, \alpha)$  is not fibered. Furthermore,  $M(-, \alpha)$  is irreducible, since the only orientable, reducible Seifert manifolds are  $S^2 \times S^1$  and  $\mathbb{R}P^3 \# \mathbb{R}P^3$  (and in particular, are closed). Finally,  $M(-, \alpha)$  is the complement of a knot in an elliptic manifold as observed in Section 2.  $\square$

*Remark 5.* Further examples may be constructed in an analogous way by removing a regular fiber from any manifold which has a Seifert structure with base orbifold  $\mathbb{R}P^2$  with any positive number of singular fibers. It is also possible to construct examples, in a similar manner, admitting Sol geometry. The main observation is that every Sol rational homology sphere is an L-space [3, Theorem 2]. Since every such L-space arises by identifying two twisted  $I$ -bundles along the boundary tori, one may consider the complement of the knot  $K_0$  in one of the twisted  $I$ -bundles. In this setting, our construction goes through almost verbatim, having noticed that the obvious essential torus must be horizontal to the purported fibration of the exterior of  $K_0$ .

*Question 6.* All of our examples have non-hyperbolic exterior. Do there exist examples of hyperbolic, non-fibered knots for which every non-longitudinal surgery is an L-space?

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